

A test of correlation among wavelet based parameter estimates in long memory processes

Un Test di Correlazione delle Stime parametriche basate su Wavelet in Processi a Memoria Lunga

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Riassunto: I modelli a memoria lunga rivestono una grande importanza per la descrizione di serie storiche osservate in diversi ambiti applicativi. Recentemente è stato proposto uno stimatore non parametrico del parametro di memoria lunga di processi stazionari basato su decomposizione wavelet, le cui proprietà statistiche sono studiate nell'ipotesi di incorrelazione dei coefficienti della trasformata wavelet. Nel presente lavoro è analizzata per numerosità finite la correlazione tra i coefficienti della trasformata wavelet e il suo impatto sulla correlazione tra stime del parametro di memoria lunga calcolate su blocchi disgiunti del processo originale. L'analisi è svolta mediante simulazione.

Keywords: long memory processes, statistical inference, wavelet decomposition

1. Introduction

Granger (1966) first pointed out the relevance of the low frequency components, embodied in the long range dependence (LRD) paradigm based on the aggregation of microunits (Taqqu *et al.*, 1997), in the estimated power spectra of many economic variables, such as industrial production and commodity price indices.

In recent years, a semiparametric estimator of the long memory parameter of stationary processes with long range dependence based on wavelet decomposition has been proposed by Veitch and Abry (1999). Such estimator is asymptotically unbiased and efficient under the idealized assumption of independence of the wavelet coefficients. Whitcher and Jensen (2000) highlighted the problem of the time-varying behaviour of the long memory parameter, for which an optimal test has been proposed by Veitch and Abry (2001), based on the idealized assumption of independence of the wavelet coefficients.

In this paper the impact of the correlation of the wavelet coefficients on the correlation among the wavelet long memory estimates is studied through the use of a permutation asymptotic test of randomness for finite sample size.

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2. The wavelet-based estimator of the long memory parameter

A discrete parameter (second order) stationary process $\{X_j; j \geq 1\}$, with variance σ^2 , is LRD if its correlation function is $r(k) \approx c_r k^{-(1-\alpha)}$ ($0 < \alpha < 1$) as k tends to infinity. The increments of a H -self similar process with (second order) stationary increments form an LRD stationary process, where the Hurst parameter H and the long memory parameter are linked by the relationship $H = (1 + \alpha)/2$ (Beran, 1994). The discrete wavelet coefficients of a function $x(t)$ are (Daubechies, 1992) $d_{jk} = \int_{-\infty}^{+\infty} x(t) \psi_{jk}(t) dt$, $j, k \in \mathbb{Z}$, where the $\psi_{jk}(t)$ are the wavelet functions, whose whole set can be derived from a single function, the mother wavelet $\psi_0(t)$, through operations of scaling and dilation/contraction. The coefficients $\{d_{jk}\}$ represent the Discrete Wavelet transform of $x(t)$. The Haar wavelet is often used for its finite support property and the simplicity of the coefficient computation procedure. In contrast to the trigonometric functions employed in the Fourier series, wavelets are well localized both in time and frequency. The wavelet coefficients of a long memory process have recently received considerable attention because of their use in the estimation of the long memory parameter α . The wavelet coefficients of a discrete parameter (second order) stationary LRD process $\{X_j; j \geq 1\}$ satisfy the property (Flandrin, 1992) that, for $N \geq (\alpha - 1)/2$, the wavelet coefficients d_{jk} with fixed scale index j form a stationary process, and, for $N \geq \alpha/2$, $E(d_{jk} d_{j'k'}) \approx |k - k'|^{\alpha-1-2N}$ goes to 0 as N and $|k - k'|$ tend to infinity (N is a positive integer, the number of vanishing moments of the wavelet, such that $\forall k = 0, 1, \dots, N-1, \int t^k \psi_0(t) dt = 0$). The linear wavelet estimator is based on the regression model $\log_2(1/n_j \sum_{k=1}^{n_j} d_{jk}^2 - g_j) = j\alpha + \log_2 c_f I(\alpha, \psi_0)$, where $I(\alpha, \psi_0)$ stands for $\int \nu^{-\alpha} |\psi_0(\nu)|^2 d\nu$, $\psi_0(\nu)$ representing the Fourier transform of the mother wavelet ψ_0 . The estimator of $(\alpha, \log_2(c_f I(\alpha, \psi_0)))$ is unbiased, asymptotically Gaussian and efficient under the idealized assumptions that i) the process $\{X_j; j \geq 1\}$ is Gaussian, ii) the random variables $\{d_{jk}\}$, j fixed, are i.i.d. (independent identically distributed), the processes $\{d_{jk}\}$ and $\{d_{j'k'}\}$, $j \neq j'$, are independent (Veitch and Abry, 1999). In Bardet *et al.* (2000) the correlation between wavelet coefficients has been taken into account.

3. Impact of the correlation of the wavelet coefficients

A simulation study has been performed to evaluate the impact of the correlation among wavelet coefficients on the correlation among the long memory linear wavelet estimates computed on non overlapping blocks of the original process. The circulant matrix method

(Davies and Harte 1987) has been employed to generate a sequence of length n , by embedding the correlation matrix S of the original process into a non negative definite matrix S' of size $m \geq 2(n-1)$ that is circulant. Two values of H have been considered, namely $H=0.6$ and $H=0.8$, for both of which a trace of stationary increments of a self similar process, of length $n=32768=2^{15}$, has been generated.

Abry and Veitch propose an optimal (uniformly most powerful) test for the time constancy of the long memory parameter over time. If the trace sequence is splitted into m non overlapping blocks and $\hat{\alpha}_j$ denotes the linear wavelet estimate of the long memory parameter in block ' j ', the null hypothesis H_0 is that the m r.v.'s $\hat{\alpha}_j$ are identically distributed (Veitch and Abry 2001). The optimality properties of the test hold under the condition of independence of the linear wavelet estimates in each block, which is motivated by the idealized assumption *ii*) of decorrelation of the wavelet coefficients. In order to test the correlation among long memory estimates each trace has been splitted into 4 and 8 non overlapping blocks of length 2^{13} and 2^{12} , respectively. In each block the long memory linear wavelet estimator has been computed. The scales used are all but the lowest one. A test of randomness based on a descriptive measure of dependence between observations, the oscillation index of order h , has been applied. The test statistic is

$$A = (n-1)^{-1} \sum_{i=1}^{n-1} f(x_{i+1}, x_i), \quad f: R^2 \rightarrow R_+ \text{ being a continuous function having the following}$$

properties: $f(x, y)=0$ iff $x=y$; $f(x, y)=f(y, x) \quad \forall (x, y) \in R^2$; for any triple x, y, z , of points such that $x < y < z$, the inequality $f(x, y) < f(x, z)$ holds. Let us consider a sequence $X_1, X_2, \dots, X_n$, of n nondegenerate r.v.'s. The null hypothesis is that they are i.i.d., against the alternative that they are not. It is intuitive to reject the null hypothesis when the observed value of A is "too small" or "too large". The main result by Conti *et al.* (1994) is that

$$\frac{\sqrt{n}(A-\eta)}{\sqrt{V+2C}} \xrightarrow{d} N(0,1) \quad n \rightarrow \infty \quad \text{where} \quad C = \text{Cov}[f(x_i, x_j), f(x_i, x_k)], \quad \eta = E[f(x_i, x_j)],$$

$V = \text{Var}[f(x_i, x_j)]$. As the quantities η , V and C are unknown, consistent estimators of them are introduced in order to build the rejection region of the test. The rejection region of the test has the following form: $|\sqrt{n}(A-\hat{\eta})| \geq z_{1-\alpha/2} \sqrt{(\hat{V}-2\hat{C})/n}$. In this paper the

permutation distribution of A has been introduced to be applied to finite sample size. If $X_1, X_2, \dots, X_n$, are i.i.d. r.v.'s, conditionally on the observed values $x_1, x_2, \dots, x_n$, the $n!$ permutations of $x_1, x_2, \dots, x_n$ have all the same probability $1/n!$. Hence the permutation distribution of A can be used to determine the rejection region so that the test has a desired significance level. The rejection region should be placed in the tails of the sample permutation distribution of A . The formal proof that the permutation conditional version of A is asymptotically equivalent to the unconditional one can be found in Hoeffding (1952).

In the following the function f is $f(x_{i+1}, x_i) = (x_{i+1} - x_i)^2$. As the value of α is constant in each trace, the test proves the independence of the $\hat{\alpha}_j$'s. The results are presented in Table 1.

For each value of H (0.6, 0.8) and each number of blocks (4, 8) the 5% and 1% thresholds have been determined considering the exact permutation distribution of A on each trace.

The 1% lower and upper thresholds, 4 blocks, are not available (n.a.) due to the low number of resulting permuted sequences ($4!=24$) which doesn't allow to achieve the required percentile order ($1/24=0.041667$).

Table 1: Test results for the independence of the long memory estimates

H	No. of blocks	Test statistic	Lower 5% threshold	Upper 5% threshold	Lower 1% threshold	Upper 1% threshold
0.6	4	$-2.69 \cdot 10^{-4}$	$-2.73 \cdot 10^{-4}$	$2.73 \cdot 10^{-4}$	n.a.	n.a.
0.8	4	$2.64 \cdot 10^{-4}$	$-2.64 \cdot 10^{-4}$	$2.64 \cdot 10^{-4}$	n.a.	n.a.
0.6	8	$-2.70 \cdot 10^{-4}$	$-5.68 \cdot 10^{-4}$	$5.81 \cdot 10^{-4}$	$-6.80 \cdot 10^{-4}$	$7.12 \cdot 10^{-4}$
0.8	8	$-2.34 \cdot 10^{-4}$	$-3.48 \cdot 10^{-4}$	$2.83 \cdot 10^{-4}$	$-4.17 \cdot 10^{-4}$	$3.11 \cdot 10^{-4}$

The analysis shows that the observed test statistic is very close to the threshold value at the 5% significance level. Moreover the test statistic is closer to the threshold in the case $H=0.8$ with respect to the analogous case $H=0.6$. The reason is that the correlation of the $\{d_{jk}\}$ increases with increasing H (Flandrin, 1992). As a conclusion, the correlation of the wavelet coefficients could explain the correlation among the linear wavelet estimates.

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